

p -ADIC LANGLANDS FUNCTORIALITY FOR THE DEFINITE UNITARY GROUP

PAUL-JAMES WHITE

ABSTRACT. We formalise a notion of p -adic Langlands functoriality for the definite unitary group. This extends the classical notion of Langlands functoriality to the setting of eigenvarieties. We apply some results of Chenevier to obtain some cases of p -adic Langlands functoriality by interpolating known cases of classical Langlands functoriality.

1. INTRODUCTION

The Langlands functoriality conjectures predict deep properties of automorphic representations of connected reductive groups. For definite unitary groups, Chenevier [Che10] has constructed a space of p -adic automorphic forms which interpolate the classical automorphic forms. The p -adic automorphic forms can be parameterised by a p -adic rigid analytic space known as the *eigenvariety*. It is a natural problem to extend the notion of Langlands functoriality to the setting of eigenvarieties, that is to introduce a notion of p -adic Langlands functoriality. Chenevier [Che05] studied this problem in the case of the Jacquet-Langlands correspondence. Chenevier was able to extend the Jacquet-Langlands correspondence to the setting of eigenvarieties. The aim of this article is to formalise a notion of p -adic Langlands functoriality for the definite unitary group and to construct some examples.

The $\overline{\mathbf{Q}}_p$ -points on an eigenvariety \mathcal{D} are parameterised by pairs (λ, κ) where $\lambda : \mathcal{H}^- \rightarrow \overline{\mathbf{Q}}_p^\times$ is a system of eigenvalues and $\kappa \in \mathcal{W}(\overline{\mathbf{Q}}_p)$ a weight coming from certain p -adic automorphic forms. In order to introduce a notion of p -adic functoriality, we construct under a technical hypothesis (see Hypothesis 3.4.1) morphisms between the Hecke algebras and the weight spaces of definite unitary groups corresponding to a given L -homomorphism. The constructions are natural generalisations of the constructions appearing in classical Langlands functoriality with one important distinction. Unlike classical Langlands functoriality, the notion of p -adic Langlands functoriality depends upon a non-canonical choice for the refinement map in addition to the chosen L -homomorphism. Different choices for the refinement map give rise to different cases of p -adic Langlands functoriality. The need to define the refinement map comes from the fact that the eigenvariety interpolates pairs of automorphic forms and accessible refinements. The need to transfer refinements between groups requires us to introduce a non-canonical map. Having made the appropriate definitions, we proceed to obtain certain cases of p -adic Langlands

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functoriality for which the classical Langlands functoriality transfer is known. The p -adic Langlands functorial transfer is obtained by interpolating the classical Langlands functorial transfer using work of Chenevier [Che05].

Let us describe the contents of this article. In Section 2, we recall the construction of the eigenvariety for the definite unitary group. In Section 3, we introduce the notion of p -adic Langlands functoriality for the definite unitary group. In Section 4, we interpolate known cases of Langlands functoriality to obtain some cases of p -adic Langlands functoriality for the definite unitary group.

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1.2. Conventions and notation. We shall normalise the Artin map from class field theory so as to send uniformizers to geometric Frobenii. The local Langlands correspondence shall be normalised as in Harris-Taylor [HT01]. Unless specified otherwise, we shall assume that representations are irreducible and admissible with complex coefficients. Parabolic induction shall always refer to unitarily normalised parabolic induction.

If k is a p -adic field, we shall fix a uniformizer ϖ_k which shall also be written as $\overline{\varpi}$. We shall fix field isomorphisms $\iota_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$ for all rational primes p .

2. THE EIGENVARIETY

We shall recall the construction of the eigenvariety for definite unitary groups. The construction is due to Buzzard [Buz07] and Chenevier [Che10] (see also Loeffler [Loe11]).

2.1. The definite unitary group. Let E/F be a totally imaginary quadratic extension of a totally real field. Let $U = U_n = U_n(E/F)$ denote the unitary group in n -variables associated to the extension E/F that is compact at infinity and quasi-split at all finite places (cf. [Whi12, §2]). Such a group exists exactly when either n is odd or $\frac{n \cdot [F:\mathbf{Q}]}{2}$ is even (cf. [Whi12, Proposition 2.1]). The local forms of U are described below.

- For all archimedean places ν of F , $U_\nu = U \times_F F_\nu$ is isomorphic to the n -variable real compact unitary group.
- For all finite places ν of F that are non-split in E , $U_\nu \simeq U_n^*(E_\nu/F_\nu)$ the n -variable quasi-split unitary group associated to the extension E_ν/F_ν .
- For all finite places ν of F that split in E , $U_\nu \simeq GL_n/F_\nu$. We warn the reader that the isomorphism is non-canonical. It essentially depends upon the choice of a place of E lying above ν (cf. [Whi12, §2]). We shall fix such an isomorphism at the finite split places and identify the two groups.

Let $\vec{n} = (n_1, \dots, n_r) \in \mathbf{N}^r$, and let $n = \sum_{i=1}^r n_i$. We shall be interested in the fibre product of the definite unitary groups

$$U_{\vec{n}} = \prod_{i=1}^r U_{n_i}$$

where we have implicitly assumed that \vec{n} is such that the individual definite unitary groups U_{n_i} exist. We shall fix an extension of $U_{\vec{n}}/F$ to a smooth group scheme $U_{\vec{n}}/\mathcal{O}_{F, S_{\text{ram}}}$ where S_{ram} is the set of archimedean places of F and the finite places of F that ramify in E .

2.2. Some subgroups. Let ν be a finite place of F . We shall recall some important subgroups of $GL_m(F_\nu)$ below.

- Let $T_\nu \subset GL_m(F_\nu)$ be the maximal split torus consisting of the diagonal matrices.
- Let $T_\nu^0 \subset T_\nu$ be the maximal compact subgroup consisting of the diagonal matrices in $GL_m(\mathcal{O}_{F_\nu})$.
- Let $T_\nu^- \subset T_\nu$ (resp. $T_\nu^{--} \subset T_\nu$) be the submonoid consisting of the elements of the form

$$\text{diag}(x_1, \dots, x_m)$$

where $\mathbf{v}(x_1) \geq \mathbf{v}(x_2) \geq \dots \geq \mathbf{v}(x_m)$ (resp. $\mathbf{v}(x_1) > \dots > \mathbf{v}(x_m)$) where $\mathbf{v}(x)$ denotes the valuation of an element $x \in F_\nu$.

- Let $T_\nu^\varpi \subset T_\nu$ (resp. $T_\nu^{\varpi, -} \subset T_\nu$) be the subgroup (resp. submonoid) consisting of the elements of the form

$$\text{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_m})$$

where $\lambda_1, \dots, \lambda_m \in \mathbf{Z}$ (resp. $\lambda_1 \geq \dots \geq \lambda_m$).

- Let $B_\nu \subset GL_m(F_\nu)$ be the Borel subgroup consisting of the upper triangular matrices.
- Let $N_\nu \subset GL_m(F_\nu)$ be the unipotent subgroup consisting of upper triangular matrices with 1 on their diagonals.
- Let $I_\nu \subset GL_m(\mathcal{O}_\nu)$ be the Iwahori subgroup consisting of the upper triangular matrices modulo ϖ .
- Let $M_\nu \subset GL_m(F_\nu)$ be the submonoid generated by I_ν and T_ν^- .

We have canonical isomorphisms $T_\nu/T_\nu^0 \simeq T_\nu^\varpi$ and $T_\nu^-/T_\nu^0 \simeq T_\nu^{\varpi, -}$. If ν splits in E , then we define the analogous subgroups of $U_{\vec{n}}(F_\nu) = GL_{n_1}(F_\nu) \times \dots \times GL_{n_r}(F_\nu)$ in the obvious way (e.g. $T_\nu = T_{GL_{n_1}, \nu} \times \dots \times T_{GL_{n_r}, \nu}$ where $T_{GL_{n_i}, \nu}$ denotes the previously defined subgroup of $GL_{n_i}(F_\nu)$ of diagonal matrices). If S is a finite set of non-archimedean places of F that split in E , then we define the analogous subgroups of $\prod_{\nu \in S} U_{\vec{n}}(F_\nu)$ in the obvious way (e.g. $T_S = \prod_{\nu \in S} T_\nu$).

2.3. The datum. An eigenvariety for $U_{\vec{n}}$ depends upon the choice of a datum (p, S, e, ϕ) whose elements are as follows.

- p is a rational prime that splits completely in E .
- S_p is the set of places of F lying above p .
- S_∞ is the set of archimedean places of F .
- S is a finite set of non-archimedean places of F containing all the places of F that are ramified in E and such that $S \cap S_p = \emptyset$.
- $K^S = \prod_{\nu \notin S} K_\nu \subset U_{\vec{n}}(\mathbf{A}_f^S)$ is the compact open subgroup such that $K_\nu = U_{\vec{n}}(\mathcal{O}_\nu)$ is maximal hyperspecial for all $\nu \notin S_p \cup S$ and $K_\nu = I_\nu$ for all $\nu \in S_p$.
- For all $\nu \in S$, e_ν is a non-trivial idempotent of the Hecke algebra $\mathcal{C}_c^\infty(U_{\vec{n}}(F_\nu), \overline{\mathbf{Q}}_p)$.

- $e = \otimes_{\nu \in S} e_\nu \otimes \mathbf{1}_{K^S}$ seen as an idempotent of the Hecke algebra $\mathcal{C}_c^\infty(U_{\vec{n}}(\mathbf{A}_f), \overline{\mathbf{Q}}_p)$ where $\mathbf{1}_{K^S}$ denotes the identity function on the compact open subgroup K^S .
- $\phi \in \overline{\mathbf{Q}}_p[T_{S_p}^-]$.

We shall identify the sets S_∞ and S_p via the field isomorphism $\iota_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$. Explicitly this bijection is given as follows.

$$\begin{aligned} \iota_p^{-1} : \text{Hom}(F, \overline{\mathbf{Q}}_p) &\rightarrow \text{Hom}(F, \mathbf{C}) \\ \nu &\mapsto \iota_p^{-1} \circ \nu \end{aligned}$$

2.4. The Hecke algebras. We shall recall the commutative Hecke algebras that are relevant to a chosen datum (p, S, e, ϕ) . The *spherical Hecke algebra* of $U_{\vec{n}}$ outside of $S \cup S_p$ is defined to be

$$\mathcal{H}_{\text{ur}} = \mathcal{C}_c^\infty(K^{S \cup S_p} \backslash U_{\vec{n}}(\mathbf{A}_f^{S \cup S_p}) / K^{S \cup S_p}, \overline{\mathbf{Q}}_p).$$

If $\nu \in S_p$, then we shall have need of the *Atkin-Lehner* sub-algebra \mathfrak{U}_ν^- of the Hecke-Iwahori algebra

$$\mathcal{A} = \mathcal{C}_c^\infty(I_\nu \backslash U_{\vec{n}}(F_\nu) / I_\nu, \overline{\mathbf{Q}}_p).$$

The $\overline{\mathbf{Q}}_p$ -subalgebra $\mathfrak{U}_\nu^- \subset \mathcal{A}$ is defined to be the sub-algebra generated by the identity functions $\mathbf{1}_{I_\nu t I_\nu}$ for all $t \in T_\nu^-$. We finish by defining the commutative $\overline{\mathbf{Q}}_p$ -algebras

$$\mathfrak{U}^- = \otimes_{\nu \in S_p} \mathfrak{U}_\nu^- \quad \text{and} \quad \mathcal{H}^- = \mathfrak{U}^- \otimes \mathcal{H}_{\text{ur}}.$$

2.5. The weight space. If $\nu \in S_p$, then one defines \mathcal{W}_ν to be the \mathbf{Q}_p -rigid analytic space that represents the functor

$$\mathcal{W}_\nu = \text{Hom}_{\text{cts-gp}}(T_\nu^0, \mathbf{G}_m^{\text{rig}}).$$

The *weight space* is defined to be the \mathbf{Q}_p -rigid analytic space

$$\mathcal{W} = \prod_{\nu \in S_p} \mathcal{W}_\nu.$$

Remark 2.1. The weight space \mathcal{W} is isomorphic to the finite disjoint union of open balls of dimension $n[F : \mathbf{Q}]$.

Let ω be a real place of F and let $\nu = \iota_p \circ \omega$ be the corresponding place above p . We have the group embedding

$$U_{\vec{n}}(F_\omega) \hookrightarrow U_{\vec{n}}(E_\omega) = (GL_{n_1} \times \cdots \times GL_{n_r})(E_\omega).$$

We shall fix $E \hookrightarrow \mathbf{C}$ a field embedding above ω . This induces an isomorphism $E_\omega \simeq \mathbf{C}$ from which we obtain the group embedding

$$U_{\vec{n}}(F_\omega) \hookrightarrow (GL_{n_1} \times \cdots \times GL_{n_r})(\mathbf{C}) \xrightarrow{\iota_p} (GL_{n_1} \times \cdots \times GL_{n_r})(\overline{F}_\nu) = U_{\vec{n}}(\overline{F}_\nu).$$

The group $U_{\vec{n}}(F_\omega)$ is compact. Its irreducible admissible representations are obtained by restriction from the finite dimensional irreducible algebraic representations of $U_{\vec{n}}(\overline{F}_\nu)$ (cf. [BC09, §6.7]). Such representations are classified by their *highest weight characters* (with respect to the Borel B_ν). If π_ω is an irreducible admissible representation of $U_{\vec{n}}(F_\omega)$, then we shall write

$$\kappa(\pi_\omega) : T_\nu \rightarrow \overline{\mathbf{Q}}_p^\times$$

for the highest weight character of π_ω . The character is of the following form.

$$\begin{aligned} \kappa(\pi_\omega) : T_{1,\nu} \times \cdots \times T_{r,\nu} &\rightarrow \overline{\mathbf{Q}}_p^\times \\ \text{diag}(x_{1,1}, \dots, x_{1,n_1}) \times \cdots \times \text{diag}(x_{r,1}, \dots, x_{r,n_r}) &\mapsto \prod_{i=1}^r \prod_{j=1}^{n_i} x_{i,j}^{k_{i,j}} \end{aligned}$$

where $k_{i,j} \in \mathbf{Z}$ and $k_{i,j} \geq k_{i,k}$ for all $1 \leq i \leq r$ and for all $1 \leq j \leq k \leq n_i$. The tuple of integers

$$\{k_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i\}$$

is called the set of *highest weights* of the representation π_ω . The highest weight is said to be *regular* if

$$k_{i,j} > k_{i,k}$$

for all $1 \leq i \leq r$ and for all $1 \leq j < k \leq n_i$.

If π_∞ is an irreducible admissible representation of $U_{\vec{n}}(\mathbf{A}_\infty)$, then π_∞ is said to have *regular highest weight* if for all $\omega|_\infty$, the highest weight of π_ω is regular. We shall write

$$\kappa(\pi_\infty) = \otimes_{\omega|_\infty} \kappa(\pi_\omega) : T_{S_p} \rightarrow \overline{\mathbf{Q}}_p^\times$$

for the *highest weight character* of π_∞ . Such a character (or more precisely its restriction to $T_{S_p}^0$) gives a $\overline{\mathbf{Q}}_p$ -valued point in the weight space

$$\kappa(\pi_\infty) \in \mathcal{W}(\overline{\mathbf{Q}}_p).$$

A point $\delta \in \mathcal{W}(\overline{\mathbf{Q}}_p)$ is said to be *regular classical* if it is of the form $\kappa(\pi_\infty)$ for some π_∞ of regular highest weight.

2.6. Refinements. We shall recall the notion of an accessible refinement following [Che10, §1.4] (see also [BC09, §6.4]). Let $\nu \in S_p$ and let π_ν be an irreducible admissible representation of $U_{\vec{n}}(F_\nu) = GL_{n_1}(F_\nu) \times \cdots \times GL_{n_r}(F_\nu)$. A *refinement* of π_ν is an unramified character

$$\chi_\nu : T_\nu^\omega \simeq T_\nu/T_\nu^0 \rightarrow \overline{\mathbf{Q}}_p^\times$$

such that π_ν appears as a constituent of the induced representation $\text{Ind}_{B_\nu}^{U_{\vec{n},\nu}} \iota_p^{-1} \chi_\nu$. A refinement is said to be *accessible* if π_ν appears as a *subrepresentation* of the induced representation.

Remark 2.2. A continuous group homomorphism $\chi_\nu : T_\nu^{\omega,-} \rightarrow \overline{\mathbf{Q}}_p^\times$ uniquely extends to a continuous group homomorphism $\chi_\nu : T_\nu^\omega \rightarrow \overline{\mathbf{Q}}_p^\times$. These two equivalent descriptions shall be used interchangeably throughout this article.

The following result shows that accessible refinements exist exactly for the representations with Iwahori-invariant vectors.

Lemma 2.3 (Bernstein, Borel, Casselman, Matsumoto). *Let $\nu \in S_p$.*

- *We have the equality $M_\nu = \sqcup_{t \in T_\nu^{\omega,-}} I_\nu t I_\nu$. Furthermore, the map*

$$\tau : M_\nu \rightarrow T_\nu^{\omega,-}$$

defined by the relation $m \in I_\nu \tau(m) I_\nu$ is a surjective multiplicative homomorphism.

- The map

$$\begin{aligned} T_{\nu}^{\bar{\omega}, -} &\rightarrow \mathfrak{U}_{\nu}^{-} \\ t &\mapsto \mathbf{1}_{I_{\nu} t I_{\nu}} \end{aligned}$$

is multiplicative, and it extends to give the $\overline{\mathbf{Q}}_p$ -algebra isomorphism

$$\overline{\mathbf{Q}}_p[T_{\nu}^{\bar{\omega}, -}] \xrightarrow{\sim} \mathfrak{U}_{\nu}^{-}.$$

- Let π_{ν} be an irreducible admissible representation of $U_{\bar{n}}(F_{\nu})$. We shall view π_{ν} as a $\overline{\mathbf{Q}}_p[T_{\nu}^{\bar{\omega}, -}]$ -module via the previous isomorphism. Then

$$(\pi_{\nu}^{I_{\nu}})^{\text{ss}} \simeq \bigoplus \iota_p^{-1} \circ (\chi_{\nu} \delta_{B_{\nu}}^{-1/2})$$

where χ_{ν} runs through the accessible refinements of π_{ν} , **ss** denotes semi-simplification, and $\delta_{B_{\nu}} : T_{\nu}/T_{\nu}^0 \rightarrow \overline{\mathbf{Q}}_p^{\times}$ denotes the modulus character viewed with coefficients in $\overline{\mathbf{Q}}_p$ via the isomorphism ι_p .

Proof. [BC09, §6.4] □

Remark 2.4. To give a refinement of an irreducible admissible representation of $U_{\bar{n}}(F_{\nu})$ with an Iwahori invariant vector is equivalent to giving an ordering of the eigenvalues of the semi-simple conjugacy class of $(GL_{n_1} \times \cdots \times GL_{n_r})(\overline{\mathbf{Q}}_p)$ associated to the geometric Frobenius via the local Langlands correspondence. In general, not all refinements will be accessible. For example, the Steinberg representation has a single accessible refinement.

If π is an automorphic representation of $U_{\bar{n}}(\mathbf{A}_F)$, then a *refinement* (resp. *accessible refinement*) of π is an unramified character

$$\chi = \otimes_{\nu \in S_p} \chi_{\nu} : T_{S_p}^{\bar{\omega}} \simeq T_{S_p}/T_{S_p}^0 \rightarrow \overline{\mathbf{Q}}_p^{\times}$$

such that χ_{ν} is a refinement (resp. accessible refinement) of the representation π_{ν} for all $\nu \in S_p$. To avoid problems of algebraicity in the construction of the eigenvariety, one normalises a refinement χ of π as follows

$$\nu(\pi, \chi) = \kappa(\pi_{\infty}) \chi \delta_{B_{S_p}}^{-1/2} : T_{S_p}^{\bar{\omega}} \rightarrow \overline{\mathbf{Q}}_p^{\times}$$

where $\kappa(\pi_{\infty})$ is viewed by restriction as a character of $T_{S_p}^{\bar{\omega}}$.

2.7. Module valued automorphic forms. We shall recall the notion of a module valued automorphic form (see [Gro99] for the general theory). Following the discussion in Section 2.5, we have the commutative diagrams

$$\begin{array}{ccc} F^{\mathbf{C}} & \xrightarrow{\quad} & \mathbf{A}_{S_p} \\ \downarrow & & \downarrow \\ \mathbf{A}_{\infty} & \xrightarrow{\quad} & \prod_{\omega|\infty} \mathbf{C} \end{array} \quad \begin{array}{ccc} U_{\bar{n}}(F)^{\mathbf{C}} & \xrightarrow{\quad} & U_{\bar{n}}(\mathbf{A}_{S_p}) \\ \downarrow & & \downarrow \\ U_{\bar{n}}(\mathbf{A}_{\infty})^{\mathbf{C}} & \xrightarrow{\quad} & \prod_{\omega|\infty} GL_{n_1} \times \cdots \times GL_{n_r}(\mathbf{C}) \end{array}$$

where F is embedded diagonally into both \mathbf{A}_{∞} and \mathbf{A}_{S_p} . An irreducible admissible representation W of $U_{\bar{n}}(\mathbf{A}_{\infty})$ is obtained via restriction from a unique irreducible algebraic representation of $\prod_{\omega|\infty} GL_{n_1} \times \cdots \times GL_{n_r}(\mathbf{C})$, which shall also be denoted by W . Using the ring isomorphism $\iota_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$, we shall view W as a representation with coefficients

in $\overline{\mathbf{Q}}_p$. The above diagram equips W with a group action of $U_{\vec{n}}(\mathbf{A}_{S_p})$. We define the $\overline{\mathbf{Q}}_p$ -vector space

$$\mathcal{A}(U_{\vec{n}}, W) = \left\{ f : U_{\vec{n}}(F) \backslash U_{\vec{n}}(\mathbf{A}_f) \rightarrow W : \begin{array}{l} f \text{ is smooth outside of } S_p, \\ f(gk_{S_p}) = k_{S_p}^{-1} f(g), \forall g \in U_{\vec{n}}(\mathbf{A}_f), \forall k_{S_p} \in I_{S_p} \end{array} \right.$$

where by f is *smooth outside of* S_p , we mean that f is invariant under right translation by some compact open subgroup of $U_{\vec{n}}(\mathbf{A}_f^{S_p})$. The monoid $U_{\vec{n}}(\mathbf{A}_f^{S_p}) \times M_{S_p}$ acts on the space as follows

$$(\gamma k_{S_p} f)(g) = k_{S_p} f(g \gamma k_{S_p}) \quad \forall \gamma \in U_{\vec{n}}(\mathbf{A}_f^{S_p}) \quad \forall k \in M_{S_p} \quad \forall f \in \mathcal{A}(U_{\vec{n}}, W).$$

This action allows us to view $\mathcal{A}(U_{\vec{n}}, W)$ as a \mathcal{H}^- -module.

The relationship between these modules and the usual notion of an automorphic representation is given by the following result.

Lemma 2.5. *There exists an isomorphism of \mathcal{H}^- -modules*

$$\mathcal{A}(U_{\vec{n}}, W) \bigotimes_{\overline{\mathbf{Q}}_p, \iota_p^{-1}} \mathbf{C} \simeq \bigoplus_{\Pi} m(\Pi) \Pi_f^{I_{S_p}}$$

where Π runs through the automorphic representations of $U_{\vec{n}}$ such that $\Pi_{\infty} \simeq W$ and $\Pi_{S_p}^{I_{S_p}} \neq 0$, and $m(\Pi)$ denotes the multiplicity of Π in the discrete automorphic spectrum of $U_{\vec{n}}$.

Proof. [Gro99, Chapter II] □

2.8. Systems of Banach modules. We shall recall Buzzard's [Buz07] property (PR) for Banach modules and the notion of a system of Banach modules following Chenevier [Che05, §1]. Let A be a commutative noetherian $\overline{\mathbf{Q}}_p$ -Banach algebra. An A -Banach module M is said to satisfy the *property (PR)* if there exists another A -Banach module M' such that $M \oplus M'$ is isomorphic (but not necessarily isometrically isomorphic) to an orthonormal A -Banach module. A *system of (PR) A -Banach modules* is defined to be a set

$$\mathbf{M} = \{M_i; \iota_i : i \in \mathbf{N}\}$$

where for all $i \in \mathbf{N}$,

- M_i is an A -Banach module satisfying the property (PR), and
- $\iota_i : M_i \rightarrow M_{i+1}$ is an A -linear compact morphism.

The corresponding inverse limit is written as $\mathbf{M}^{\dagger} = \varprojlim M_i$. Let \mathcal{S} be a reduced separated $\overline{\mathbf{Q}}_p$ -rigid analytic space. A *sheaf of (PR) Banach modules* on \mathcal{S} is a sheaf of modules B on \mathcal{S} such that

- for all open affinoids $V \subset \mathcal{S}$, $B(V)$ is an $\mathcal{O}(V)$ -Banach module satisfying the property (PR), and
- for all open affinoids $V' \subset V \subset \mathcal{S}$, the base change morphism

$$B(V) \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(V') \rightarrow B(V')$$

is an isomorphism of $\mathcal{O}(V')$ -Banach modules.

A system of (PR) Banach modules on \mathcal{S} is a set

$$\mathbf{M} = \{M_i; \iota_i : i \in \mathbf{N}\}$$

where for all $i \in \mathbf{N}$,

- M_i is a sheaf of (PR) Banach modules on \mathcal{S} , and
- $\iota_i : M_i \rightarrow M_{i+1}$ is a sheaf morphism such that for all open affinoids $V \subset \mathcal{S}$, $\iota_i(V) : M_i(V) \rightarrow M_{i+1}(V)$ is an $\mathcal{O}(V)$ -linear compact morphism.

If $V \subset \mathcal{S}$ is an open affinoid, then we shall denote the corresponding system of (PR) $\mathcal{O}(V)$ -Banach modules by

$$\mathbf{M}(V) = \{M_i(V); \iota_i : i \in \mathbf{N}\}.$$

If $x \in \mathcal{S}(\overline{\mathbf{Q}}_p)$, then we shall denote the corresponding system of (PR) $\overline{\mathbf{Q}}_p$ -Banach modules by

$$\mathbf{M}_x = \{M_{i,x}; \iota_i : i \in \mathbf{N}\}$$

where for $i \in \mathbf{N}$, $M_{i,x} = M_i(V) \hat{\otimes}_{\mathcal{O}(V)} \overline{\mathbf{Q}}_p$ where $V \subset \mathcal{S}$ is an open affinoid neighbourhood of x and the map $\mathcal{O}(V) \rightarrow \overline{\mathbf{Q}}_p$ is the one given by the point x . An *endomorphism* of \mathbf{M} is a set

$$\phi = \{\phi_i(V) : V \subset \mathcal{S} \text{ open affinoid, } i \in \mathbf{N} \text{ sufficiently large}\}$$

where each $\phi_i(V) : M_i(V) \rightarrow M_i(V)$ is a continuous $\mathcal{O}(V)$ -linear endomorphism, and the $\phi_i(V)$ commute with both the ι_i and the base change morphisms between open affinoids $V \subset V' \subset \mathcal{S}$.

Let $\text{Comp}(\mathbf{M}) \subset \text{End}(\mathbf{M})$ be the two-sided ideal consisting of endomorphisms ϕ such that for all open affinoids $V \subset \mathcal{S}$, there exists for i sufficiently large, a continuous $\mathcal{O}(V)$ -linear morphism $\psi_i(V) : M_{i+1}(V) \rightarrow M_i(V)$ such that the following diagram commutes.

$$\begin{array}{ccc} M_i(V) & \xrightarrow{\phi_i(V)} & M_i(V) \\ \downarrow \iota_i(V) & \nearrow \psi_i(V) & \downarrow \iota_i(V) \\ M_{i+1}(V) & \xrightarrow{\phi_{i+1}(V)} & M_{i+1}(V) \end{array}$$

2.9. Buzzard's eigenvariety machine. We shall consider a datum $(\mathcal{S}, \mathbf{M}, T, \phi)$ where

- \mathcal{S} is a reduced separated $\overline{\mathbf{Q}}_p$ -rigid analytic space,
- \mathbf{M} is a system of (PR) Banach modules on \mathcal{S} ,
- T is a commutative $\overline{\mathbf{Q}}_p$ -algebra equipped with a homomorphism $T \rightarrow \text{End}(\mathbf{M})$, and
- $\phi \in T$ acts compactly on \mathbf{M} (that is ϕ acts compactly on the $M_i(V)$ for all $i \in \mathbf{N}$ and for all open affinoids $V \subset \mathcal{S}$) and satisfies the compatibility condition $\phi \in \text{Comp}(\mathbf{M})$.

A $\overline{\mathbf{Q}}_p$ -valued *system of eigenvalues* for $(\mathcal{S}, \mathbf{M}, T, \phi)$ is a pair (λ, x) where

- $x \in \mathcal{S}(\overline{\mathbf{Q}}_p)$ and
- $\lambda : T \rightarrow \overline{\mathbf{Q}}_p$ is a $\overline{\mathbf{Q}}_p$ -algebra homomorphism such that there exists a non-zero $m \in \mathbf{M}_x^\dagger$ for which $\alpha(m) = \lambda(\alpha) \cdot m$ for all $\alpha \in T$.

A system of eigenvalues (λ, x) is said to be ϕ -finite if $\lambda(\phi) \neq 0$.

Theorem 2.6. *There exists a tuple $(\mathcal{D}, \psi, \kappa)$ where*

- \mathcal{D} is a $\overline{\mathbf{Q}}_p$ -rigid analytic space,
- $\psi : T \rightarrow \mathcal{O}(\mathcal{D})$ is a $\overline{\mathbf{Q}}_p$ -algebra homomorphism, and
- $\kappa : \mathcal{D} \rightarrow \mathcal{S}$ is a morphism of rigid analytic spaces

such that

- the map $\nu = (\kappa, \psi(\phi)^{-1}) : \mathcal{D} \rightarrow \mathcal{S} \times \mathbf{G}_m^{\text{rig}}$ is a finite morphism,
- for all open affinoids $V \subset \mathcal{S} \times \mathbf{G}_m^{\text{rig}}$, the natural map

$$\psi \hat{\otimes} \nu^* : T \hat{\otimes} \mathcal{O}(V) \rightarrow \mathcal{O}(\nu^{-1}(V))$$

is surjective, and

- the natural evaluation map

$$\mathcal{D}(\overline{\mathbf{Q}}_p) \rightarrow \text{Hom}(T, \overline{\mathbf{Q}}_p), \quad x \mapsto \psi_x : (h \mapsto \psi(h)(x))$$

induces a bijection $x \mapsto (\psi_x, \kappa(x))$ between the set of $\overline{\mathbf{Q}}_p$ -valued points of \mathcal{D} and the set of ϕ -finite $\overline{\mathbf{Q}}_p$ -valued systems of eigenvalues for $(\mathcal{S}, \mathbf{M}, T, \phi)$.

Furthermore if \mathcal{D} is reduced, then $(\mathcal{D}, \psi, \kappa)$ is uniquely determined by these properties.

Proof. The eigenvariety was constructed by Buzzard [Buz07, §5] generalising an earlier construction of Coleman-Mazur [CM98]. The uniqueness, a formal property, was obtained by Bellaïche-Chenevier [BC09, Proposition 7.2.8]. \square

2.10. p -adic forms of type (p, S, e, ϕ) . If W is an irreducible admissible representation of $U_{\vec{n}}(\mathbf{A}_{\infty})$, then we define the \mathcal{H}^- -module

$$S_{\kappa(W)}^{\text{cl}} = e\mathcal{A}(U_{\vec{n}}, W).$$

Chenevier [Che10, §2] has constructed a system of (PR) Banach modules on the weight space \mathcal{W} that interpolate the spaces $S_{\kappa(W)}^{\text{cl}}$. The *system of p -adic forms* of type (p, S, e, ϕ) , denoted $\mathbf{S} = \{S_i; \iota_i : i \in \mathbf{N}\}$, satisfies the following properties.

- There exists a $\overline{\mathbf{Q}}_p$ -algebra homomorphism $\mathcal{H}^- \rightarrow \text{End}(\mathbf{S})$.
- If $t \in \overline{\mathbf{Q}}_p[T_{S_p}^{--}]$, then t viewed as an element of \mathcal{H}^- acts compactly on \mathbf{S} and satisfies the compatibility condition $t \in \text{Comp}(\mathbf{S})$.
- For all W , there exists a natural embedding of \mathcal{H}^- -modules,

$$S_{\kappa(W)}^{\text{cl}} \otimes \kappa(W) \hookrightarrow \mathbf{S}_{\kappa(W)}^{\dagger}$$

where $\kappa(W) : T_{S_p} \rightarrow \overline{\mathbf{Q}}_p^{\times}$ is viewed by restriction as a character of $T_{S_p}^{\overline{\omega}, -}$. Chenevier [Che10, Proposition 2.17] also obtains a *small slope is classical* type result. This allows us to deduce that certain p -adic forms $f \in \mathbf{S}_{\kappa(W)}^{\dagger}$ are classical, that is f lies in the image of the above embedding.

2.11. The eigenvariety of type (p, S, e, ϕ) . Let π be an automorphic representation of $U_{\vec{n}}(\mathbf{A}_F)$ such that $e(\pi_f) \neq 0$. The complex vector space $(\pi_f^{S \cup S_p})^{K^{S \cup S_p}}$ is 1-dimensional and \mathcal{H}_{ur} acts on this space via scalar multiplication. We shall write

$$\psi_{\text{ur}}(\pi) : \mathcal{H}_{\text{ur}} \rightarrow \overline{\mathbf{Q}}_p$$

for the corresponding homomorphism composed with ι_p . Let

$$\mathcal{Z} \subset \mathrm{Hom}(\mathcal{H}^-, \overline{\mathbf{Q}}_p) \times \mathcal{W}(\overline{\mathbf{Q}}_p)$$

denote the subset of pairs $(\nu(\pi, \chi) \otimes \psi_{\mathrm{ur}}(\pi), \kappa(\pi_\infty))$ where π runs through the automorphic representations of the above form and χ the accessible refinements of π .

Theorem 2.7. *There exists a unique tuple $(\mathcal{D}, \psi, \kappa, Z)$ where*

- \mathcal{D} is a reduced rigid analytic space over $\overline{\mathbf{Q}}_p$,
- $\psi : \mathcal{H}^- \rightarrow \mathcal{O}(\mathcal{D})$ is a $\overline{\mathbf{Q}}_p$ -algebra homomorphism,
- $\kappa : \mathcal{D} \rightarrow \mathcal{W}$ is a morphism of rigid analytic spaces, and
- $Z \subset \mathcal{D}(\overline{\mathbf{Q}}_p)$ is an accumulation and Zariski-dense subset (cf. [BC09, §3.3.1])

such that

- the map $\nu = (\kappa, \psi(\phi)^{-1}) : \mathcal{D} \rightarrow \mathcal{W} \times \mathbf{G}_m^{\mathrm{rig}}$ is a finite morphism,
- for all open affinoids $V \subset \mathcal{W} \times \mathbf{G}_m^{\mathrm{rig}}$, the natural map

$$\psi \hat{\otimes} \nu^* : \mathcal{H}^- \hat{\otimes} \mathcal{O}(V) \rightarrow \mathcal{O}(\nu^{-1}(V))$$

is surjective, and

- the natural evaluation map $\mathcal{D}(\overline{\mathbf{Q}}_p) \rightarrow \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}^-, \overline{\mathbf{Q}}_p)$

$$x \mapsto \psi_x : (h \mapsto \psi(h)(x))$$

induces a bijection $Z \xrightarrow{\sim} \mathcal{Z}$, $z \mapsto (\psi_z, \kappa(z))$.

In addition, it also satisfies the following properties.

- \mathcal{D} is equidimensional of dimension $\dim(\mathcal{W}) = n[F : \mathbf{Q}]$. More precisely, \mathcal{D} has a canonical admissible covering which is given by the open affinoids $\Omega \subset \mathcal{D}$ such that $\kappa(\Omega)$ is an open affinoid and the morphism $\kappa|_{\Omega} : \Omega \rightarrow \kappa(\Omega)$ is finite and surjective when restricted to each irreducible component of Ω . Furthermore, the image by κ of each irreducible component of \mathcal{D} is Zariski-open in \mathcal{W} .
- $\psi(\mathcal{H}_{\mathrm{ur}}) \subset \mathcal{O}(\mathcal{D})^{\leq 1}$ where $\mathcal{O}(\mathcal{D})^{\leq 1} \subset \mathcal{O}(\mathcal{D})$ denotes the subring of functions bounded by 1.

Proof. The eigenvariety is constructed by applying the eigenvariety machine to the p -adic forms of type (p, S, e, ϕ) (cf. [Che10, Theorem 1.6]). We remark that the density of classical points follows from a small slope is classical type result (cf. [Che10, Proposition 2.17]) whilst the fact that \mathcal{D} is reduced follows from a result of Chenevier [Che05, Proposition 3.9]. \square

3. p -ADIC LANGLANDS FUNCTORIALITY: DEFINITIONS

We shall generalise the notion of Langlands functoriality to the setting of eigenvarieties. The setup is as follows. Let $H = U_{\vec{n}}$, and $G = U_m$ where $\vec{n} = (n_1, \dots, n_r)$ and $m \in \mathbf{N}$. Let (p, S, e_H, ϕ_H) and (p, S, e_G, ϕ_G) be data. (Concerning notation, we shall add a subscript H or G to previously defined objects to indicate the group to which they are associated.) Let

$$\xi : {}^L H \rightarrow {}^L G$$

be an L -homomorphism. Recall that ${}^L H = \widehat{H} \rtimes W_F$ where $\widehat{H} = GL_{n_1} \times \cdots \times GL_{n_r}(\mathbf{C})$ and the Weil group W_F acts via projection onto $\text{Gal}(E/F) = \{1, c\}$ where c acts via the isomorphism

$$\begin{aligned} c : GL_{n_1} \times \cdots \times GL_{n_r}(\mathbf{C}) &\rightarrow GL_{n_1} \times \cdots \times GL_{n_r}(\mathbf{C}) \\ g_1 \times \cdots \times g_r &\mapsto \Phi_{n_1}^{-1} g_1^{-1} \Phi_{n_1}^{-1} \times \cdots \times \Phi_{n_r}^{-1} g_r^{-1} \Phi_{n_r}^{-1} \end{aligned}$$

where

$$\Phi_t = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{t-1} & \cdots & 0 & 0 \end{pmatrix}.$$

3.1. Unramified places. If $\nu \notin S$ is a non-archimedean place of F , then H_ν and G_ν are unramified groups and ξ induces a map from the $K_{H,\nu}$ -unramified representations of $H(F_\nu)$ to the $K_{G,\nu}$ -unramified representations of $G(F_\nu)$. Dual to this transfer, there exists a morphism of spherical Hecke algebras (cf. [Mín11, §2.7])

$$\xi_{\text{ur},\nu}^* : \mathcal{C}_c^\infty(K_{G,\nu} \backslash G(F_\nu) / K_{G,\nu}, \overline{\mathbf{Q}}_p) \rightarrow \mathcal{C}_c^\infty(K_{H,\nu} \backslash H(F_\nu) / K_{H,\nu}, \overline{\mathbf{Q}}_p).$$

Combining these morphisms, we obtain the morphism of spherical Hecke algebras

$$\xi_{\text{ur}}^* = \otimes_{\nu \notin S \cup S_p} \xi_{\text{ur},\nu}^* : \mathcal{H}_{G,\text{ur}} \rightarrow \mathcal{H}_{H,\text{ur}}.$$

3.2. Places in S_p . Let $\nu \in S_p$. We shall slightly modify the construction of the morphism of the spherical Hecke algebras (cf. [Mín11, §2]) to obtain a map of refinements.

The $\overline{\mathbf{Q}}_p$ -valued characters of $T_{H,\nu}/T_{H,\nu}^0$ (resp. $T_{G,\nu}/T_{G,\nu}^0$) are naturally parameterised by the $\overline{\mathbf{Q}}_p$ -valued points of $\widehat{T}_{H,\nu}$ (resp. $\widehat{T}_{G,\nu}$) where $\widehat{T}_{H,\nu}$ (resp. $\widehat{T}_{G,\nu}$) denotes the dual torus of $T_{H,\nu}$ (resp. $T_{G,\nu}$). This is seen via the following chain of canonical bijections

$$\begin{aligned} \widehat{T}_{H,\nu}(\overline{\mathbf{Q}}_p) &= \text{Hom}(X^*(\widehat{T}_{H,\nu}), \overline{\mathbf{Q}}_p^\times) \\ &= \text{Hom}(X_*(T_{H,\nu}), \overline{\mathbf{Q}}_p^\times) \\ &= \text{Hom}(T_{H,\nu}/T_{H,\nu}^0, \overline{\mathbf{Q}}_p^\times) \\ &= \text{Hom}(T_{H,\nu}^\vee, \overline{\mathbf{Q}}_p^\times) \end{aligned}$$

where X^* (resp. X_*) denotes the group of algebraic characters (resp. algebraic co-characters) of the corresponding algebraic torus. The first bijection is simply the definition of the $\overline{\mathbf{Q}}_p$ -points of $\widehat{T}_{H,\nu}$. The second bijection follows from the canonical bijection $X^*(\widehat{T}_{H,\nu}) = X_*(T_{H,\nu})$. The third bijection is induced from the canonical bijection

$$\begin{aligned} X_*(T_{H,\nu}) &\rightarrow T_{H,\nu}/T_{H,\nu}^0 \\ \alpha^\vee &\mapsto \alpha^\vee(\overline{\omega}). \end{aligned}$$

The fourth bijection follows from the canonical bijection $T_{H,\nu}/T_{H,\nu}^0 \simeq T_{H,\nu}^\vee$.

The L -homomorphism $\xi : {}^L H \rightarrow {}^L G$ restricts to give a map $\xi : \widehat{H} \times \mathcal{F}_\nu \rightarrow \widehat{G} \times \mathcal{F}_\nu$ where \mathcal{F}_ν refers to the geometric Frobenius element of W_{F_ν} . This induces a map

$$\widehat{T}_{H,\nu}(\overline{\mathbf{Q}}_p) \rightarrow \widehat{T}_{G,\nu}(\overline{\mathbf{Q}}_p).$$

This map is canonical up to composing with an isomorphism of the form

$$\begin{aligned} \hat{T}_{G,\nu}(\overline{\mathbf{Q}}_p) &\rightarrow \hat{T}_{G,\nu}(\overline{\mathbf{Q}}_p) \\ \text{diag}(x_1, \dots, x_m) &\mapsto \text{diag}(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \end{aligned}$$

where $\sigma \in \mathfrak{S}_m$ is a permutation.

Dual to the map $\hat{T}_{H,\nu}(\overline{\mathbf{Q}}_p) \rightarrow \hat{T}_{G,\nu}(\overline{\mathbf{Q}}_p)$, we have a morphism of $\overline{\mathbf{Q}}_p$ -algebras

$$\mathcal{R}_\nu : \overline{\mathbf{Q}}_p[T_{G,\nu}^\omega] \rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}^\omega]$$

which is canonical up to precomposing with an isomorphism of the form

$$\begin{aligned} \iota_\sigma : T_{G,\nu} &\rightarrow T_{G,\nu} \\ \text{diag}(x_1, \dots, x_m) &\mapsto \text{diag}(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \end{aligned}$$

where $\sigma \in \mathfrak{S}_m$ is a permutation. We shall refer to \mathcal{R}_ν as the *refinement map*.

Remark 3.1. The reason that the morphism \mathcal{R}_ν is non-canonical is due to the fact that we are working at the level of refinements whilst the classical Langlands correspondence operates at the level of representations. Consequently in order to obtain a map of refinements, one is obliged to specify an ordering of the refinement map. Two such maps differ by ι_σ for a choice of $\sigma \in \mathfrak{S}_m$.

Lemma 3.2. *Let $\pi_{H,\nu}$ and $\pi_{G,\nu}$ be irreducible admissible representations of $H(F_\nu)$ and $G(F_\nu)$ respectively. Assume that $\pi_{H,\nu}^{I_{H,\nu}} \neq 0$, $\pi_{G,\nu}^{I_{G,\nu}} \neq 0$, and $\pi_{G,\nu}$ is the Langlands functorial transfer of $\pi_{H,\nu}$ via the L -homomorphism ξ . If $\chi_{H,\nu} : T_{H,\nu}^\omega \simeq T_{H,\nu}^0/T_{H,\nu}^0 \rightarrow \overline{\mathbf{Q}}_p^\times$ is a refinement of $\pi_{H,\nu}$, then $\chi_{H,\nu} \circ \mathcal{R}_\nu : T_{G,\nu}^\omega \simeq T_{G,\nu}^0/T_{G,\nu}^0 \rightarrow \overline{\mathbf{Q}}_p^\times$ is a refinement of $\pi_{G,\nu}$.*

Remark 3.3. We stress that there is no claim that accessible refinements are mapped to accessible refinements. Such a claim would be false in general.

Proof. Writing as follows the unramified character

$$\chi_{H,\nu} = \chi_{H,1,\nu} \times \cdots \times \chi_{H,r,\nu} : T_{GL_{n_1},\nu}^\omega \times \cdots \times T_{GL_{n_r},\nu}^\omega \rightarrow \overline{\mathbf{Q}}_p^\times,$$

we observe that the semi-simple conjugacy class of $GL_{n_1} \times \cdots \times GL_{n_r}(\overline{\mathbf{Q}}_p)$ corresponding to the geometric Frobenius via the local Langlands correspondence for $\pi_{H,\nu}$ is represented by the element

$$\text{diag}(\chi_{H,1,\nu}(\overline{\omega}, 1, \dots, 1), \dots, \chi_{H,1,\nu}(1, \dots, 1, \overline{\omega})) \times \cdots \times \text{diag}(\chi_{H,r,\nu}(\overline{\omega}, 1, \dots, 1), \dots, \chi_{H,r,\nu}(1, \dots, 1, \overline{\omega})).$$

Our construction of the morphism \mathcal{R}_ν is based upon a slight modification of the construction of the corresponding morphism of the spherical Hecke algebras $\xi_{\text{ur},\nu}^*$ (cf. [Mín11, §2]). It follows that the semi-simple conjugacy class of $GL_m(\overline{\mathbf{Q}}_p)$ corresponding to the geometric Frobenius via the local Langlands correspondence for $\pi_{G,\nu}$ is represented by the element

$$\text{diag}(\chi_{G,\nu}(\overline{\omega}, 1, \dots, 1), \dots, \chi_{G,\nu}(1, \dots, 1, \overline{\omega}))$$

where $\chi_{G,\nu} = \chi_{H,\nu} \circ \mathcal{R}_\nu : T_{GL_m,\nu}^\omega \rightarrow \overline{\mathbf{Q}}_p^\times$. That is $\chi_{G,\nu}$ is a refinement of $\pi_{G,\nu}$. \square

It will be convenient for reasons of algebraicity arising in the construction of the eigenvariety to renormalise the morphism as follows

$$\begin{aligned} \mathcal{R}'_\nu : \overline{\mathbf{Q}}_p[T_{G,\nu}^\omega] &\rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}^\omega] \\ t &\mapsto \mathcal{R}_\nu(t) \cdot \delta_{B_{G,\nu}}^{-1/2}(t) \cdot (\delta_{B_{H,\nu}}^{1/2} \circ \mathcal{R}_\nu)(t). \end{aligned}$$

The effect of the normalisation is described by the following trivial lemma.

Lemma 3.4. *If $\chi_{H,\nu} : T_{H,\nu}^\omega \rightarrow \overline{\mathbf{Q}}_p^\times$ is a homomorphism, then the following maps are equal*

$$(\chi_{H,\nu} \cdot \delta_{B_{H,\nu}}^{-1/2}) \circ \mathcal{R}'_\nu = (\chi_{H,\nu} \circ \mathcal{R}_\nu) \cdot \delta_{B_{G,\nu}}^{-1/2} : T_{G,\nu}^\omega \rightarrow \overline{\mathbf{Q}}_p^\times.$$

3.3. Weight space map. It remains to study the behaviour at archimedean places. We shall assume that our L -homomorphism ξ satisfies the following hypothesis, which includes a compatibility condition between the behaviour of ξ at the non-archimedean places in S_p and the archimedean places.

Hypothesis 3.4.1. *If $\nu \in S_p$, then there exist $\overline{\mathbf{Q}}_p$ -algebra homomorphisms*

$$\begin{aligned} \Xi_{\mathcal{W},\nu}^* : \overline{\mathbf{Q}}_p[T_{G,\nu}] &\rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}] \\ \Lambda_\nu^* : \overline{\mathbf{Q}}_p[T_{G,\nu}^{\omega,-}] &\rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}^{\omega,-}] \end{aligned}$$

such that the following conditions are satisfied.

- Let $\omega = \iota_p^{-1}\nu$ be the archimedean place corresponding to ν . Let $\pi_{H,\omega}$, and $\pi_{G,\omega}$ be irreducible admissible representations of $H(F_\omega)$ and $G(F_\omega)$ respectively such that $\pi_{G,\omega}$ corresponds to $\pi_{H,\omega}$ via Langlands functoriality for ξ . Then there exists a $\sigma \in \mathfrak{S}_m$ (depending upon $\pi_{G,\omega}$ and $\pi_{H,\omega}$) such that the following two characters are equal

$$\kappa(\pi_{G,\omega}) = \kappa(\pi_{H,\omega}) \circ \Xi_{\mathcal{W},\nu}^* \circ \iota_\sigma : T_{G,\nu} \rightarrow \overline{\mathbf{Q}}_p^\times.$$

Furthermore $\Xi_{\mathcal{W},\nu}^*$ restricts to give a \mathbf{Z}_p -algebra homomorphism

$$\Xi_{\mathcal{W},\nu}^* : \mathbf{Z}_p[T_{G,\nu}^0] \rightarrow \mathbf{Z}_p[T_{H,\nu}^0].$$

- $\Xi_{\mathcal{W},\nu}^*$ restricts to give a $\overline{\mathbf{Q}}_p$ -algebra homomorphism

$$\Xi_{\mathcal{W},\nu}^* : \overline{\mathbf{Q}}_p[T_{G,\nu}^\omega] \rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}^\omega]$$

such that for all homomorphisms $\chi, \delta : T_{H,\nu}^{\omega,-} \rightarrow \overline{\mathbf{Q}}_p^\times$,

$$(\chi \cdot \delta) \circ \Lambda_\nu^* = (\chi \circ \mathcal{R}'_\nu) \cdot (\delta \circ \Xi_{\mathcal{W},\nu}^*) : T_{G,\nu}^{\omega,-} \rightarrow \overline{\mathbf{Q}}_p^\times.$$

If $\nu \in S_p$, then the morphism $\Xi_{\mathcal{W},\nu}^*$ induces a \mathbf{Z}_p -algebra homomorphism

$$\Xi_{\mathcal{W},\nu}^* : \mathbf{Z}_p[[T_{G,\nu}^0]] \rightarrow \mathbf{Z}_p[[T_{H,\nu}^0]].$$

This induces a \mathbf{Q}_p -rigid analytic morphism (cf. [dJ95, §7])

$$\Xi_{\mathcal{W},\nu} : \mathcal{W}_{H,\nu} \rightarrow \mathcal{W}_{G,\nu}.$$

Together these morphisms induce a \mathbf{Q}_p -rigid analytic morphism of the weight spaces

$$\Xi_{\mathcal{W}} = \prod_{\nu \in S_p} \Xi_{\mathcal{W},\nu} : \mathcal{W}_H \rightarrow \mathcal{W}_G.$$

We also note that the morphisms $\Lambda_\nu^* : \overline{\mathbf{Q}}_p[T_{G,\nu}^{\overline{\omega},-}] \rightarrow \overline{\mathbf{Q}}_p[T_{H,\nu}^{\overline{\omega},-}]$ where $\nu \in S_p$ induce morphisms of the Atkin-Lehner algebras

$$\begin{aligned} \Lambda_\nu^* &: \mathfrak{U}_{G,\nu}^- \rightarrow \mathfrak{U}_{H,\nu}^- \\ \Lambda^* &= \otimes_{\nu \in S_p} \Lambda_\nu^* : \mathfrak{U}_G^- \rightarrow \mathfrak{U}_H^-. \end{aligned}$$

3.4. Definitions. We are now in a position to introduce a notion of p -adic functoriality.

Definition 3.5. A rigid analytic morphism

$$\Xi : \mathcal{D}_H \rightarrow \mathcal{D}_G$$

is said to be the p -adic Langlands functoriality morphism for the tuple $(\xi, \Xi_{\mathcal{W}}, \Lambda^*)$ (or simply ξ if the context is clear) if the following diagrams commute.

$$\begin{array}{ccc} \mathcal{D}_H & \xrightarrow{\Xi} & \mathcal{D}_G \\ \downarrow \kappa_H & & \downarrow \kappa_G \\ \mathcal{W}_H & \xrightarrow{\Xi_{\mathcal{W}}} & \mathcal{W}_G \end{array} \quad \begin{array}{ccc} \mathcal{H}_G^- & \xrightarrow{\Lambda^* \otimes \xi_{\text{ur}}^*} & \mathcal{H}_H^- \\ \downarrow \psi_G & & \downarrow \psi_H \\ \mathcal{O}(\mathcal{D}_G) & \xrightarrow{\Xi^*} & \mathcal{O}(\mathcal{D}_H) \end{array}$$

Lemma 3.6. *If the rigid analytic morphism Ξ exists, then it is unique.*

Proof. Since the eigenvarieties \mathcal{D}_H and \mathcal{D}_G are reduced, the morphism Ξ is completely determined by the induced map on the $\overline{\mathbf{Q}}_p$ -valued points. The morphisms $\Xi_{\mathcal{W}}$ and $\Lambda^* \otimes \xi_{\text{ur}}^*$ induce a map

$$\text{Hom}(\mathcal{H}_H^-, \overline{\mathbf{Q}}_p) \times \mathcal{W}_H(\overline{\mathbf{Q}}_p) \rightarrow \text{Hom}(\mathcal{H}_G^-, \overline{\mathbf{Q}}_p) \times \mathcal{W}_G(\overline{\mathbf{Q}}_p)$$

which fixes the map of the eigenvarieties on the $\overline{\mathbf{Q}}_p$ -valued points. The result follows. \square

Remark 3.7. Let us emphasise the main difference between classical Langlands functoriality and p -adic Langlands functoriality. In the p -adic setting, one must make a non-canonical choice for the map of refinements \mathcal{R}_ν for $\nu \in S_p$, and different choices of maps give rise to different notions of functoriality.

Concerning compatibility with classical Langlands functoriality, we have the following result.

Lemma 3.8. *Assume that $\Xi : \mathcal{D}_H \rightarrow \mathcal{D}_G$, the p -adic Langlands functoriality morphism for ξ , exists. Let π_H (resp. π_G) be an automorphic representations of H (resp. G) such that $e_H(\pi_{H,f}) \neq 0$ (resp. $e_G(\pi_{G,f}) \neq 0$) equipped with an accessible refinement χ_H (resp. χ_G). Let $x_H \in \mathcal{D}_H(\overline{\mathbf{Q}}_p)$ (resp. $x_G \in \mathcal{D}_G(\overline{\mathbf{Q}}_p)$) be the $\overline{\mathbf{Q}}_p$ -point on the eigenvariety corresponding to the pair $(\nu_H(\pi_H, \chi_H) \otimes \psi_{H,\text{ur}}(\pi_H), \kappa_H(\pi_{H,\infty}))$ (resp. $(\nu_G(\pi_G, \chi_G) \otimes \psi_{G,\text{ur}}(\pi_G), \kappa_G(\pi_{G,\infty}))$). Then the following statements hold.*

- *If $\Xi(x_H) = x_G$, then π_G corresponds to π_H via Langlands functoriality for ξ at both the archimedean places and the non-archimedean places $\nu \notin S \cup S_p$.*
- *Conversely, if*
 - *π_G corresponds to π_H via Langlands functoriality for ξ at the non-archimedean places $\nu \notin S \cup S_p$;*
 - *$\Xi_{\mathcal{W}}(\kappa(\pi_{H,\infty})) = \kappa(\pi_{G,\infty}) \in \mathcal{W}_G(\overline{\mathbf{Q}}_p)$; and*

– for all $\nu \in S_p$, $\chi_{H,\nu} \circ \mathcal{R}_\nu = \chi_{G,\nu} : T_{G,\nu}^\omega \rightarrow \overline{\mathbf{Q}}_p^\times$,

then $\Xi(x_H) = x_G$.

Proof. We shall prove the first statement since the proof of the second statement follows from the definitions in a similar fashion. The conditions imposed upon π_H and π_G ensure that the points x_H and x_G exist. The representation π_G is seen to be the transfer of π_H via Langlands functoriality at both the archimedean places and the non-archimedean places $\nu \notin S \cup S_p$. The former follows from our conditions imposed upon the weight space morphism $\Xi_{\mathcal{W}}$ whilst the later follows from the properties of the spherical Hecke algebra morphism $\xi_{\text{ur}}^* : \mathcal{H}_{G,\text{ur}} \rightarrow \mathcal{H}_{H,\text{ur}}$. \square

4. p -ADIC LANGLANDS FUNCTORIALITY: CONSTRUCTION OF THE MORPHISM

We shall construct the p -adic Langlands functoriality morphism when the L -homomorphism ξ is the Langlands direct sum.

4.1. The Langlands direct sum for the unitary group. We shall define the Langlands direct sum L -homomorphism and make explicit the induced transfer of representations. Let E/F be a totally imaginary quadratic extension of a totally real field. Let $\eta : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbf{C}^\times$ be the character associated to the field extension E/F via class field theory. Fix a unitary character

$$\mu : E^\times \backslash \mathbf{A}_E^\times \rightarrow \mathbf{C}^\times$$

such that

- μ extends η ;
- if $\nu \in S_p$, then μ is unramified at ν ; and
- if ν is non-archimedean and inert in E , then μ is unramified at ν .

The character μ can be seen via class field theory as a character of the Weil group W_E . At archimedean places ω of E , the Hecke character is of the form (cf. [BC09, §6.9.2])

$$\begin{aligned} \mu_\omega : \mathbf{C}^\times &\rightarrow \mathbf{C}^\times \\ z &\mapsto (z/\bar{z})^{\alpha_\omega} \end{aligned}$$

for some half-integer α_ω .

For all integers $i \in \mathbf{Z}$, we shall define the Hecke character $\mu_i : E^\times \backslash \mathbf{A}_E^\times \rightarrow \mathbf{C}^\times$

$$\mu_i = \begin{cases} \mu & \text{if } i \text{ is odd} \\ \mathbf{1} & \text{if } i \text{ is even} \end{cases}$$

and correspondingly for all archimedean places ω of E , we define

$$\alpha_{i,\omega} = \begin{cases} \alpha_\omega & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

In what follows, we shall often abuse notation and view an archimedean place ω of F as the corresponding archimedean place of E lying above ω and vice versa. We shall also often view the characters μ_i as having values in $\overline{\mathbf{Q}}_p$ via the isomorphism ι_p .

We can now introduce the following L -homomorphism, which we shall refer to as the *Langlands direct sum L -homomorphism*.

$$\begin{aligned} \xi = \xi_{\vec{n}} : {}^L(U_{n_1} \times \cdots \times U_{n_r}) &\rightarrow {}^L U_n \\ g_1 \times \cdots \times g_r \times 1 &\mapsto \text{diag}(g_1, \dots, g_r) \times 1 \\ I_{n_1} \times \cdots \times I_{n_r} \times w &\mapsto \text{diag}(\mu_{n-n_1}(w) I_{n_1}, \dots, \mu_{n-n_r}(w) I_{n_r}) \times w \quad \forall w \in W_E \\ I_{n_1} \times \cdots \times I_{n_r} \times w_c &\mapsto \text{diag}(\Phi_{n_1}, \dots, \Phi_{n_r}) \Phi_n^{-1} \times w_c \end{aligned}$$

where w_c denotes a chosen lift of $c \in \text{Gal}(E/F)$ and I_{n_i} denotes the identity matrix of $GL_{n_i}(\mathbf{C})$.

Remark 4.1. If $\vec{n} = (a, b)$, then $\xi_{\vec{n}}$ is the endoscopic L -homomorphism studied in [Rog90, §4].

We shall now make the corresponding Langlands functorial transfer of representations explicit in some cases (cf. [Whi12, §4.2]).

- Assume that $\omega|_{\infty}$. Let π_{ω} be an irreducible admissible representation of $U_{\vec{n}}(F_{\omega})$. Let $\{k_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i\}$ be the ordered set of highest weights of π_{ω} . The Langlands functorial transfer of π_{ω} is defined if the numbers

$$k_{i,j} + \frac{n_i + 1}{2} - j + \alpha_{n-n_i, \omega}, \quad i = 1, \dots, r \quad j = 1, \dots, n_i$$

are distinct (otherwise the corresponding L -parameter of $U_n(F_{\omega})$ will not be relevant). Let

$$k'_i, \quad i = 1, \dots, n$$

denote the reordering of the $k_{i,j} + \frac{n_i + 1}{2} - j + \alpha_{n-n_i, \omega}$ such that $k'_s > k'_t$ for all $s > t$, and let $k_i = k'_i - \frac{n+1}{2} + i$ for all $i = 1, \dots, n$. The Langlands functorial transfer of π_{ω} , denoted $\xi_{\vec{n}}(\pi_{\omega})$, is the irreducible admissible representation of $U_n(F_{\omega})$ whose highest weights are equal to

$$\{k_i : i = 1, \dots, n\}$$

We remind the reader that the terms $\frac{n_i + 1}{2} - j$ and $-\frac{n+1}{2} + i$ appear due to the difference between the Langlands parameterisation and the highest weight parameterisation of an irreducible admissible representation of $U(F_{\omega})$ (cf. [BC09, §6.7]).

- Assume that ν is a finite place of F that splits in E . If π_{ν} is an irreducible admissible unitary representation of $U_{\vec{n}}(F_{\nu}) = GL_{n_1} \times \cdots \times GL_{n_r}(F_{\nu})$, then

$$\text{Ind}_{P_{\vec{n}}}^{U_n} \pi_{1,\nu} \cdot \mu_{n-n_1,\nu} \times \cdots \times \pi_{r,\nu} \cdot \mu_{n-n_r,\nu}$$

where $P_{\vec{n}}$ denotes the standard Parabolic subgroup with Levi-component $U_{\vec{n}}$, is an irreducible admissible unitary representation of $U_n(F_{\nu})$ (cf. [Ber84]), and it is the Langlands functorial transfer of π_{ν} .

- Assume that ν is a finite place of F that remains inert in E . If π_{ν} is unramified, then the correspondence can be explicitly described in terms of Satake parameters (cf. [Mín11, §4]).

Let $\nu \in S_p$. We shall study the map on refinements induced by \mathcal{R}_ν . Explicitly, the morphism can be seen to be equal to $\mathcal{R}_\nu = \mathcal{R}_{0,\nu} \circ \iota_{\sigma_\nu}$ for a choice of $\sigma_\nu \in \mathfrak{S}_n$ where

$$\begin{aligned} \mathcal{R}_{0,\nu} : \overline{\mathbf{Q}}_p[T_{U_{n,\nu}}^\omega] &\rightarrow \overline{\mathbf{Q}}_p[T_{U_{\tilde{n},\nu}}^\omega] \\ \text{diag}(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{r,n_r}) &\mapsto \mu_{n-n_1,\nu}(x_{1,1} \cdots x_{1,n_1}) \cdot \text{diag}(x_{1,1}, \dots, x_{1,n_1}) \times \\ &\quad \cdots \times \mu_{n-n_r,\nu}(x_{r,1} \cdots x_{r,n_r}) \cdot \text{diag}(x_{r,1}, \dots, x_{r,n_r}). \end{aligned}$$

The $\overline{\mathbf{Q}}_p$ -algebra morphism $\mathcal{R}_{0,\nu}$ induces a map on refinements, which can be explicitly described as follows. Let

$$\chi_{U_{\tilde{n},\nu}} = \chi_{U_{\tilde{n},1,\nu}} \times \cdots \times \chi_{U_{\tilde{n},r,\nu}} : T_{U_{\tilde{n},\nu}}^\omega \simeq T_{U_{\tilde{n},\nu}}^0 / T_{U_{\tilde{n},\nu}}^0 \rightarrow \overline{\mathbf{Q}}_p^\times$$

be an unramified character. Then

$$\begin{aligned} \chi_{U_{\tilde{n},\nu}} \circ \mathcal{R}_{0,\nu} : T_{U_{n,\nu}}^0 / T_{U_{n,\nu}}^0 &\rightarrow \overline{\mathbf{Q}}_p^\times \\ \text{diag}(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{r,n_r}) &\mapsto \prod_{i=1}^r \mu_{n-n_i,\nu}(x_{i,1} \cdots x_{i,n_i}) \cdot \chi_{U_{\tilde{n},i,\nu}}(\text{diag}(x_{i,1}, \dots, x_{i,n_i})). \end{aligned}$$

Under certain conditions, one can ensure that accessible refinements are mapped to accessible refinements, which will not be the case in general.

Lemma 4.2. *Let $\nu \in S_p$. Let $\pi_{U_{\tilde{n},\nu}} = \times_{i=1}^r \pi_{U_{\tilde{n},i,\nu}}$ be an irreducible admissible tempered representation of $U_{\tilde{n}}(F_\nu)$ such that $\pi_{U_{\tilde{n},\nu}}^{I_{U_{\tilde{n},\nu}}} \neq 0$. Let $\chi_{U_{\tilde{n},\nu}}$ be an accessible refinement of $\pi_{U_{\tilde{n},\nu}}$. Let $\pi_{U_{n,\nu}}$ be the $\xi_{\tilde{n}}$ -Langlands functorial transfer of $\pi_{U_{\tilde{n},\nu}}$ to $U_n(F_\nu)$. Let $\sigma_\nu \in \mathfrak{S}_n$ such that*

$$\sigma_\nu(i) < \sigma_\nu(j) \text{ for all } n_0 + \cdots + n_t \leq i < j \leq n_{t+1} \text{ for all } 0 \leq t \leq r-1$$

where $n_0 = 0$. Then $\pi_{U_{n,\nu}}^{I_{U_{n,\nu}}} \neq 0$ and $\chi_{U_{n,\nu}} \circ \mathcal{R}_{0,\nu} \circ \iota_{\sigma_\nu}$ is an accessible refinement of $\pi_{U_{n,\nu}}$.

Proof. Since the representation $\pi_{U_{\tilde{n},\nu}}$ is tempered with an Iwahori-invariant vector, it is isomorphic to a representation of the form

$$\pi_{U_{\tilde{n},\nu}} \simeq \times_{i=1}^r \text{Ind}_{P_i}^{GL_{n_i}} \gamma_{i,1} \cdot \text{St}(d_{i,1}) \times \cdots \times \gamma_{i,m_i} \cdot \text{St}(d_{i,m_i})$$

where for all $i = 1, \dots, r$

- $\sum_{j=1}^{m_i} d_{i,j} = n_i$;
- M_i is the standard Levi subgroup with blocks of size $d_{i,j}$ and P_i is the corresponding standard Parabolic; and
- for all $j = 1, \dots, m_i$,
 - if $d_{i,j} \neq 1$, then $\gamma_{i,j}$ is an unramified unitary character and $\text{St}(d_{i,j})$ denotes the standard Steinberg representation of $GL_{d_{i,j}}(F_\nu)$ and
 - if $d_{i,j} = 1$, then $\gamma_{i,j} \neq \mathbf{1}$ is a non-trivial unramified unitary character and $\text{St}(1) = \mathbf{1}$ denotes the trivial character.

It follows that

$$\pi_{U_{n,\nu}} \simeq \text{Ind}_P^{GL_n} \times_{i=1}^r \mu_{n-n_i,\nu} \times_{j=1}^{m_i} \gamma_{i,j} \cdot \text{St}(d_{i,j})$$

where P denotes the corresponding standard Parabolic. This representation has an Iwahori invariant vector. The fact that the morphism \mathcal{R}_ν , under the restrictions imposed

upon σ_ν , sends an accessible refinement of $\pi_{U_{\vec{n}},\nu}$ to an accessible refinement of $\pi_{U_n,\nu}$ follows from a general result of Zelevinsky [Zel80, Theorem 6.1]. \square

Let $\nu \in S_p$ and let $\omega = \iota_p^{-1} \circ \nu \in S_\infty$ denote the corresponding archimedean place. Choose a $\sigma_\nu \in \mathfrak{S}_n$ as in Lemma 4.2. The morphisms Λ_ν^* and $\Xi_{\mathcal{W},\nu}^*$ can be chosen to be $\Lambda_\nu^* = \Lambda_{0,\nu}^* \circ \iota_{\sigma_\nu}$ and $\Xi_{\mathcal{W},\nu}^* = \Xi_{\mathcal{W},0,\nu}^* \circ \iota_{\sigma_\nu}$ where

$$\begin{aligned} \Xi_{\mathcal{W},0,\nu}^* : \overline{\mathbf{Q}}_p[T_{U_n,\nu}] &\rightarrow \overline{\mathbf{Q}}_p[T_{U_{\vec{n}},\nu}] \\ \text{diag}(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{r,n_r}) &\mapsto \prod_{i=1}^r \prod_{j=1}^{n_i} x_{i,j}^{\alpha_{n-n_i,\omega} + \frac{n_i-n}{2} + n_1 + \dots + n_{i-1}} \text{diag}(x_{i,1}, \dots, x_{i,n_i}) \\ \Lambda_{0,\nu}^* : \overline{\mathbf{Q}}_p[T_{U_n,\nu}^{\bar{\omega},-}] &\rightarrow \overline{\mathbf{Q}}_p[T_{U_{\vec{n}},\nu}^{\bar{\omega},-}] \\ t &\mapsto \Lambda_{0,\nu}^\dagger(t) \cdot \delta_{B_{U_n,\nu}}^{-1/2}(t) \cdot (\delta_{B_{U_{\vec{n}},\nu}}^{1/2} \circ \Lambda_{0,\nu}^\dagger)(t) \\ \Lambda_{0,\nu}^\dagger : \overline{\mathbf{Q}}_p[T_{U_n,\nu}^{\bar{\omega},-}] &\rightarrow \overline{\mathbf{Q}}_p[T_{U_{\vec{n}},\nu}^{\bar{\omega},-}] \\ \text{diag}(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{r,n_r}) &\mapsto \prod_{i=1}^r \prod_{j=1}^{n_i} \mu_{n-n_i,\nu}(x_{i,1} \cdots x_{i,n_i}) \\ &\quad x_{i,j}^{\alpha_{n-n_i,\omega} + \frac{n_i-n}{2} + n_1 + \dots + n_{i-1}} \text{diag}(x_{i,1}, \dots, x_{i,n_i}). \end{aligned}$$

Lemma 4.3. *The induced morphism*

$$\Xi_{\mathcal{W}} : \mathcal{W}_{U_{\vec{n}}} \rightarrow \mathcal{W}_{U_n}$$

is an isomorphism of weight spaces.

Proof. This follows directly from the fact that the induced \mathbf{Z}_p -algebra morphisms

$$\Xi_{\mathcal{W},\nu}^* : \mathbf{Z}_p[[T_{U_n,\nu}^0]] \rightarrow \mathbf{Z}_p[[T_{U_{\vec{n}},\nu}^0]]$$

are isomorphisms for all $\nu \in S_p$. \square

Lemma 4.4. *Let $\nu \in S_p$ and let $\sigma_\nu \in \mathfrak{S}_n$ be as in Lemma 4.2 (this fixes the morphism $\mathcal{R}_\nu = \mathcal{R}_{0,\nu} \circ \iota_{\sigma_\nu}$). The morphisms Λ_ν^* and $\Xi_{\mathcal{W},\nu}^*$ are the unique morphisms that satisfy Hypothesis 3.4.1 for this choice of \mathcal{R}_ν .*

Proof. The fact that the morphisms satisfy Hypothesis 3.4.1 follows from a simple check. To see that the morphisms are unique, we observe the following. By the first condition of Hypothesis 3.4.1, one observes that the morphism $\Xi_{\mathcal{W},\nu}^* = \Xi_{\mathcal{W},0,\nu}^* \circ \iota_{\sigma_\nu}$ is uniquely determined up to composing with an isomorphism of the form $\iota_{\sigma'}$ where $\sigma' \in \mathfrak{S}_n$. The second condition of Hypothesis 3.4.1 forces σ' to be the identity permutation, that is the morphism $\Xi_{\mathcal{W},\nu}^*$ is the unique morphism to satisfy Hypothesis 3.4.1. The uniqueness of the morphisms Λ_ν^* is seen to follow from the second condition of Hypothesis 3.4.1. \square

4.2. p -adic Langlands functoriality for the Langlands direct sum. For all $\nu \in S_p$, let $\sigma_\nu \in \mathfrak{S}_n$ as in Lemma 4.2. We shall consider data $(p, S, e_{U_{\vec{n}}}, \phi_{U_{\vec{n}}})$ and $(p, S, e_{U_n}, \phi_{U_n})$ of the following form.

- E/F is a totally imaginary quadratic extension of a totally real field that is unramified at all finite places.

- $\lambda \notin S_p$ is a non-archimedean place of F that does not split in E .
- S is a finite set of places of F such that $\lambda \in S$, $S \cap S_p = \emptyset$, and if $\nu \in S - \lambda$ then ν is non-archimedean and splits in E .
- For all $\nu \in S - \lambda$, $e_{U_{\bar{n}}, \nu} = \mathbf{1}_{I_{U_{\bar{n}}, \nu}}$ (resp. $e_{U_n, \nu} = \mathbf{1}_{I_{U_n, \nu}}$) seen as an idempotent of the Hecke algebra $\mathbf{C}_c^\infty(U_{\bar{n}}(F_\nu), \overline{\mathbf{Q}}_p)$ (resp. $\mathbf{C}_c^\infty(U_n(F_\nu), \overline{\mathbf{Q}}_p)$).
- $e_{U_{\bar{n}}, \lambda}$ corresponds to the Bernstein component of a supercuspidal representation (cf. [BC09, Example 7.3.3]) $\sigma_{U_{\bar{n}}, \lambda} = \sigma_{U_{\bar{n}}, 1, \lambda} \times \cdots \times \sigma_{U_{\bar{n}}, r, \lambda}$ such that for all $i \neq j$,

$$\sigma_{U_{\bar{n}}, i, \lambda} \not\sim \sigma_{U_{\bar{n}}, j, \lambda}.$$

$e_{U_n, \lambda}$ is the sum of the idempotents corresponding to the Bernstein components containing the discrete series representations $\sigma_{U_n, \lambda} \in \Pi$ where Π is the L -packet of discrete series representations of $U_n(F_\lambda)$ which are the $\xi_{\bar{n}}$ -Langlands functorial transfer of $\sigma_{U_{\bar{n}}, \lambda}$ (see [Whi12, §3.3.5] for a discussion of the local Langlands classification of discrete series representations for the quasi-split unitary group due to Mœglin [Mœg07]).

- $\phi_{U_{\bar{n}}} \in \overline{\mathbf{Q}}_p[T_{U_{\bar{n}}, S_p}^-]$ and $\phi_{U_n} = \Lambda^*(\phi_{U_{\bar{n}}})$ which due to our explicit description of Λ^* we see lies in $\overline{\mathbf{Q}}_p[T_{U_n, S_p}^-]$.

We shall now construct the p -adic Langlands functoriality morphism $\Xi : \mathcal{D}_{U_{\bar{n}}} \rightarrow \mathcal{D}_{U_n}$ for the Langlands direct sum $\xi_{\bar{n}}$ and our choice of σ_ν for $\nu \in S_p$. In order to do so, we must first introduce an auxiliary eigenvariety.

4.2.1. *Auxiliary eigenvariety.* The morphism

$$\Lambda^* \otimes \xi_{\text{ur}}^* : \mathcal{H}_{U_n}^- \rightarrow \mathcal{H}_{U_{\bar{n}}}^-$$

equips the p -adic forms of type $(p, S, e_{U_{\bar{n}}}, \phi_{U_{\bar{n}}})$, denoted $\mathbf{S}_{U_{\bar{n}}} = \{S_{U_{\bar{n}}, i}; \iota_i : i \in \mathbf{N}\}$, with an action of the Hecke algebra $\mathcal{H}_{U_n}^-$

$$\mathcal{H}_{U_n}^- \rightarrow \mathcal{H}_{U_{\bar{n}}}^- \rightarrow \text{End}(\mathbf{S}_{U_{\bar{n}}})$$

where the second morphism is the natural action of the Hecke algebra on $\mathbf{S}_{U_{\bar{n}}}$. Feeding the datum $(\mathcal{W}_{U_{\bar{n}}}, \mathbf{S}_{U_{\bar{n}}}, \mathcal{H}_{U_{\bar{n}}}^-, \phi_{U_{\bar{n}}})$ into Buzzard's eigenvariety machine, we obtain the corresponding Eigenvariety $(\mathcal{D}', \psi', \kappa')$ which is seen to be reduced (cf. [Che05, Proposition 3.9]).

4.2.2. *The two morphisms.* We shall construct the morphism $\Xi : \mathcal{D}_{U_{\bar{n}}} \rightarrow \mathcal{D}_{U_n}$ as the composite of two morphisms Ξ_1 and Ξ_2 where $\Xi_1 : \mathcal{D}_{U_{\bar{n}}} \rightarrow \mathcal{D}'$ and $\Xi_2 : \mathcal{D}' \rightarrow \mathcal{D}_{U_n}$ are $\overline{\mathbf{Q}}_p$ -rigid analytic morphisms such that the following diagrams commute.

$$\begin{array}{ccc} \mathcal{D}_{U_{\bar{n}}} & \xrightarrow{\Xi_1} & \mathcal{D}' \\ \downarrow \kappa_{U_{\bar{n}}} & \searrow \kappa' & \\ \mathcal{W}_{U_{\bar{n}}} & & \end{array} \quad \begin{array}{ccc} \mathcal{H}_{U_n}^- & \xrightarrow{\Lambda^* \otimes \xi_{\text{ur}}^*} & \mathcal{H}_{U_{\bar{n}}}^- \\ \downarrow \psi' & & \downarrow \psi_{U_{\bar{n}}} \\ \mathcal{O}(\mathcal{D}') & \xrightarrow{\Xi_1^*} & \mathcal{O}(\mathcal{D}_{U_{\bar{n}}}) \end{array}$$

$$\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{\Xi_2} & \mathcal{D}_{U_n} \\
\downarrow \kappa' & & \downarrow \kappa_{U_n} \\
\mathcal{W}_{U_{\bar{n}}} & \xrightarrow{\Xi_{\mathcal{W}}} & \mathcal{W}_{U_n}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{H}_{U_n}^- & & \\
\downarrow \psi_{U_n} & \searrow \psi' & \\
\mathcal{O}(\mathcal{D}_{U_n}) & \xrightarrow{\Xi_2^*} & \mathcal{O}(\mathcal{D}')
\end{array}$$

4.2.3. Existence of the morphism Ξ_1 .

Lemma 4.5. *There exists a $\overline{\mathbf{Q}}_p$ -rigid analytic morphism $\Xi_1 : \mathcal{D}_{U_{\bar{n}}} \rightarrow \mathcal{D}'$ for which the respective diagrams in Section 4.2.1 commute.*

Proof. In order to construct Ξ_1 , we shall require Buzzard's construction of the eigenvariety which we shall briefly recall. Details of this construction can be found in either [BC09, §7.3.6] or [Buz07].

One associates to $\phi_{U_{\bar{n}}}$ a unique Fredholm power series $P_{\phi_{U_{\bar{n}}}}(T) \in 1 + T \cdot \mathcal{O}(\mathcal{W}_{U_{\bar{n}}})\{\{T\}\}$ such that for all open affinoids $V \subset \mathcal{W}_{U_{\bar{n}}}$ and for all $i \in \mathbf{N}$ sufficiently large

$$P_{\phi_{U_{\bar{n}}}}(T)|_V = \det(1 - T \cdot \phi_{U_{\bar{n}}} |_{S_{U_{\bar{n}},i}(V)}) \in 1 + T \cdot \mathcal{O}(V)\{\{T\}\}.$$

Let $Z(P_{\phi_{U_{\bar{n}}}}) \subset \mathcal{W}_{U_{\bar{n}}} \times \mathbf{G}_m^{\text{rig}}$ denote the associated Fredholm hypersurface. The eigenvariety $\mathcal{D}_{U_{\bar{n}}}$ is constructed as a cover for $Z(P_{\phi_{U_{\bar{n}}}})$. We shall write $\text{pr}_1 : Z(P_{\phi_{U_{\bar{n}}}}) \rightarrow \mathcal{W}_{U_{\bar{n}}}$ for the projection map onto the $\mathcal{W}_{U_{\bar{n}}}$ component. There exists a canonical admissible covering $\mathcal{C}_{U_{\bar{n}}}^*$ of $Z(P_{\phi_{U_{\bar{n}}}})$ given by the open affinoids $\Omega^* \subset Z(P_{\phi_{U_{\bar{n}}}})$ for which the image $\text{pr}_1(\Omega^*) \subset \mathcal{W}_{U_{\bar{n}}}$ is an open affinoid and the induced map $\text{pr}_1|_{\Omega^*} : \Omega^* \rightarrow \text{pr}_1(\Omega^*)$ is finite. If $\Omega^* \in \mathcal{C}_{U_{\bar{n}}}^*$ and $V = \text{pr}_1(\Omega^*)$, then one can use the resultant to canonically factorize

$$P_{\phi_{U_{\bar{n}}}}|_V = Q_{\phi_{U_{\bar{n}}}} R_{\phi_{U_{\bar{n}}}} \in \mathcal{O}(V)\{\{T\}\}.$$

This induces for $i \in \mathbf{N}$ sufficiently large, an $\mathcal{O}(V)$ -Banach module $\mathcal{H}_{U_{\bar{n}}}^-$ -equivariant decomposition

$$S_{U_{\bar{n}},i}(V) = S_{U_{\bar{n}}}(\Omega^*) \oplus N(\Omega^*, i)$$

where $S_{U_{\bar{n}}}(\Omega^*)$ is a finite projective $\mathcal{O}(V)$ -module that is independent of i . The part of the eigenvariety lying above Ω^* , that is $\nu_{U_{\bar{n}}}^{-1}(\Omega^*)$ is equal to the maximal spectrum of the image of $\mathcal{H}_{U_{\bar{n}}}^-$ in $\text{End}(S_{U_{\bar{n}}}(\Omega^*))$.

Consider now the analogous objects used in the construction of the eigenvariety \mathcal{D}' . By construction, the underlying Banach spaces $S_{U_{\bar{n}},i}(V)$ are identical to those used in the construction of $\mathcal{D}_{U_{\bar{n}}}$ and the corresponding action of ϕ_{U_n} is equal to the action of $\phi_{U_{\bar{n}}}$. It follows that the corresponding Fredholm power series are equal and the associated Fredholm hypersurfaces are identical.

Let $\Omega^* \in \mathcal{C}_{U_{\bar{n}}}^*$, let $V = \text{pr}_1(\Omega^*)$, and let $i \in \mathbf{N}$ be sufficiently large. Using the obvious notation since the Fredholm power series are identical, we see that $\Omega^* \in \mathcal{C}'$ and

$$S_{U_{\bar{n}}}(\Omega^*) = S'(\Omega^*).$$

Write A'_{Ω^*} (resp. $A_{U_{\bar{n}},\Omega^*}$) for the image of $\mathcal{H}_{U_n}^-$ (resp. $\mathcal{H}_{U_{\bar{n}}}^-$) in $\text{End}(S_{U_{\bar{n}}}(\Omega^*))$. Since $\mathcal{H}_{U_n}^-$ acts through the morphism $\Lambda^* \otimes \xi_{\text{ur}}^* : \mathcal{H}_{U_n}^- \rightarrow \mathcal{H}_{U_{\bar{n}}}^-$, we have a natural embedding

$A'_{\Omega^*} \hookrightarrow A_{U_{\bar{n}}, \Omega^*}$. This induces a morphism between the parts of the eigenvariety lying above Ω^* :

$$\nu_{U_{\bar{n}}}^{-1}(\Omega^*) \rightarrow \nu'^{-1}(\Omega^*).$$

These morphisms glue together to give a morphism $\Xi_1 : \mathcal{D}_{U_{\bar{n}}} \rightarrow \mathcal{D}'$, which is seen to make the respective diagrams in Section 4.2.1 commute. \square

4.2.4. *Existence of the morphism Ξ_2 .* The morphism Ξ_2 shall be constructed by applying some work of Chenevier [Che05] to interpolate the classical Langlands functoriality transfer for the Langlands direct sum.

Lemma 4.6. *Let $\pi_{U_{\bar{n}}}$ be an automorphic representation of $U_{\bar{n}}(\mathbf{A})$ such that*

- *for all archimedean places ν , $\pi_{U_{\bar{n}}, \nu}$ has regular highest weight and the local ξ -Langlands functorial transfer to $U_n(F_\nu)$ exists and has regular highest weight,*
- *for all non-archimedean places $\nu \neq \lambda$, $\pi_{U_{\bar{n}}, \nu}^{K_{U_{\bar{n}}, \nu}} \neq 0$, and*
- *$\pi_{U_{\bar{n}}, \lambda} \simeq \sigma_{U_{\bar{n}}, \lambda}$.*

Then,

- *$\pi_{U_{\bar{n}}}$ appears in the discrete automorphic spectrum of $U_{\bar{n}}(\mathbf{A})$ with multiplicity one,*
- *$\pi_{U_{\bar{n}}, \nu}$ is tempered for all non-archimedean places ν that split in E ,*
- *there exists an automorphic representation π_{U_n} of $U_n(\mathbf{A})$ such that π_{U_n} is the $\xi_{\bar{n}}$ -Langlands functorial transfer of $\pi_{U_{\bar{n}}}$ at all places, and*
- *there exists a constant $C \in \mathbf{N}$ depending only upon $\sigma_{U_{\bar{n}}, \lambda}$, such that $\dim e_{U_{\bar{n}}}(\pi_{U_{\bar{n}}, f}) \leq C \cdot \dim e_{U_n}(\pi_{U_n, f})$.*

Proof. Let us assume the terminology of [Whi12]. The multiplicity one statement is a special case of [Whi12, Theorem 11.2]. Write $\pi_{U_{\bar{n}}} = \pi_{U_{\bar{n}}, 1} \times \cdots \times \pi_{U_{\bar{n}}, r}$. For $i = 1, \dots, r$, let Π_i be the Langlands base change of $\pi_{U_{\bar{n}}, i}$ to $GL_{n_i}(\mathbf{A}_E)$ which exists as a cuspidal automorphic representation (cf. [Whi12, Theorem 6.1]). If $\nu = vv'$ is a non-archimedean place of F that splits in E , then by the definition of local Langlands base change

$$\Pi_{i, \nu} = \Pi_{i, v} \times \Pi_{i, v'} \simeq \pi_{U_{\bar{n}}, i, \nu} \times \pi_{U_{\bar{n}}, i, \nu}^\vee \quad \forall i = 1, \dots, r.$$

By a result of Shin [Shi11, Corollary 1.3], the representations Π_i are tempered at all finite places. It follows that $\pi_{U_{\bar{n}}, \nu}$ is tempered. To see the existence of π_{U_n} , we first define the automorphic representation of $GL_n(\mathbf{A}_E)$

$$\Pi' = \mu_{n-n_1} \Pi_1 \boxplus \cdots \boxplus \mu_{n-n_r} \Pi_r.$$

Let π_{U_n} be an automorphic representation of $U_n(\mathbf{A})$ whose Langlands base change is isomorphic to Π' whose existence is guaranteed by [Whi12, Corollary 11.3]. One checks from the respective definitions that π_{U_n} is the $\xi_{\bar{n}}$ -Langlands functorial transfer of $\pi_{U_{\bar{n}}}$ at all places.

For the final statement, define

$$C = \max \left(\left\lceil \frac{\dim e_{U_{\bar{n}}, \lambda}(\sigma_{U_{\bar{n}}, \lambda})}{\dim e_{U_n, \lambda}(\sigma_{U_n, \lambda})} \right\rceil : \sigma_{U_n, \lambda} \in \Pi \right)$$

where Π denotes the L -packet of discrete series representations of $U_n(F_\lambda)$ which are the Langlands functorial transfer of $\sigma_{U_{\bar{n}}, \lambda}$. We observe that $\dim e_{U_{\bar{n}}}(\pi_{U_{\bar{n}}, f}) = \prod_{\nu \in S \cup S_p} \dim e_{U_{\bar{n}}, \nu}(\pi_{U_{\bar{n}}, \nu})$ and $\dim e_{U_n}(\pi_{U_n, f}) = \prod_{\nu \in S \cup S_p} \dim e_{U_n, \nu}(\pi_{U_n, \nu})$. If $\nu \in S \cup S_p - \lambda$, then $\dim e_{U_{\bar{n}}, \nu}(\pi_{U_{\bar{n}}, \nu}) =$

$\dim \pi_{U_{\tilde{n}}, \nu}^{I_{U_{\tilde{n}}, \nu}}$ (resp. $\dim e_{U_n, \nu}(\pi_{U_n, \nu}) = \dim \pi_{U_n, \nu}^{I_{U_n, \nu}}$) which is equal to the number of accessible refinements of $\pi_{U_{\tilde{n}}, \nu}$ (resp. $\pi_{U_n, \nu}$) (see Lemma 2.3 whose result trivially extends to $\nu \in S \cup S_p - \lambda$). It follows from Lemma 4.2 (trivially extended to $\nu \in S \cup S_p - \lambda$) that

$$\dim e_{U_{\tilde{n}}, \nu}(\pi_{U_{\tilde{n}}, \nu}) \leq \dim e_{U_n, \nu}(\pi_{U_n, \nu}).$$

The result follows. \square

Lemma 4.7. *There exists a $\overline{\mathbf{Q}}_p$ -rigid analytic morphism $\Xi_2 : \mathcal{D}' \rightarrow \mathcal{D}_{U_n}$ for which the respective diagrams in Section 4.2.1 commute.*

Proof. We remind the reader that $\Xi_{\mathcal{W}} : \mathcal{W}_{U_{\tilde{n}}} \rightarrow \mathcal{W}_{U_n}$ is an isomorphism which allows us to identify the two spaces (cf. Lemma 4.3). The existence of such a Ξ_2 will follow from the interpolation result of Chenevier [Che05, Theorem 1] combined with a small slope is classical type result [Che10, Proposition 2.17] upon confirmation that for all irreducible admissible representations $W_{U_{\tilde{n}}}$ of $U_{\tilde{n}}(\mathbf{A}_{\infty})$ of regular highest weight, for all irreducible admissible representations W_{U_n} of $U_n(\mathbf{A}_{\infty})$ of regular highest weight such that $\Xi_{\mathcal{W}}(\delta(W_{U_{\tilde{n}}})) = \delta_{W_{U_n}}$, and for all $h \in \mathcal{H}_{U_n}^-$, we have that

$$\det(1 - Th \cdot \phi_{U_n} | S_{U_{\tilde{n}}, \kappa(W_{U_{\tilde{n}}})}^{\text{cl}} \otimes \kappa(W_{U_{\tilde{n}}})) | \det(1 - Th \cdot \phi_{U_n} | S_{U_n, \kappa(W_{U_n})}^{\text{cl}} \otimes \kappa(W_{U_n}))$$

as elements of $\mathbf{C}_p[T]$ where we are using the morphism $\Lambda^* \otimes \xi_{\text{ur}}^* : \mathcal{H}_{U_n}^- \rightarrow \mathcal{H}_{U_{\tilde{n}}}^-$ to view $S_{U_{\tilde{n}}, \kappa(W_{U_{\tilde{n}}})}^{\text{cl}}$ as a $\mathcal{H}_{U_n}^-$ -module. In fact, the result will follow from a slightly weaker statement namely that there exists a $C \in \mathbf{N}$ such that for all $W_{U_{\tilde{n}}}$, W_{U_n} , and h as above, we have that

$$\det(1 - Th \cdot \phi_{U_n} | S_{U_{\tilde{n}}, \kappa(W_{U_{\tilde{n}}})}^{\text{cl}} \otimes \kappa(W_{U_{\tilde{n}}})) | \det(1 - Th \cdot \phi_{U_n} | S_{U_n, \kappa(W_{U_n})}^{\text{cl}} \otimes \kappa(W_{U_n}))^C.$$

The fact that this weaker statement suffices follows from the fact that the eigenvariety \mathcal{D}_{U_n} is seen to be canonically isomorphic to the eigenvariety constructed from the data

$$(\mathcal{W}_{U_n}, \mathbf{S}_{U_n}^{\oplus C}, \mathcal{H}_{U_n}^-, \phi_{U_n})$$

where the direct sum of system of (PR) $\overline{\mathbf{Q}}_p$ -Banach modules over \mathcal{W}_{U_n} is defined in the logical way.

The desired statement is a simple consequence of Lemma 4.2 and Lemma 4.6, which we shall now explain. By Lemma 2.5, we have the following isomorphisms of $\mathcal{H}_{U_n}^-$ -modules,

$$\begin{aligned} S_{U_n, \kappa(W_{U_n})}^{\text{cl}} &\simeq \bigoplus_{\pi_{U_n}} m(\pi_{U_n}) e_{U_n}(\Pi_{U_n, f}) \otimes_{\mathbf{C}, \iota_p} \overline{\mathbf{Q}}_p \\ S_{U_{\tilde{n}}, \kappa(W_{U_{\tilde{n}}})}^{\text{cl}} &\simeq \bigoplus_{\pi_{U_{\tilde{n}}}} m(\pi_{U_{\tilde{n}}}) e_{U_{\tilde{n}}}(\Pi_{U_{\tilde{n}}, f}) \otimes_{\mathbf{C}, \iota_p} \overline{\mathbf{Q}}_p \end{aligned}$$

where π_{U_n} (resp. $\pi_{U_{\tilde{n}}}$) runs through the automorphic representations of $U_n(\mathbf{A})$ (resp. $U_{\tilde{n}}(\mathbf{A})$) such that $\pi_{U_n, \infty} \simeq W_{U_n}$ (resp. $\pi_{U_{\tilde{n}}, \infty} \simeq W_{U_{\tilde{n}}}$) and $e_{U_n}(\Pi_{U_n, f}) \neq 0$ (resp. $e_{U_{\tilde{n}}}(\Pi_{U_{\tilde{n}}, f}) \neq 0$), and $\pi_{U_{\tilde{n}}}$ is viewed as a $\mathcal{H}_{U_n}^-$ -module via the morphism $\Lambda^* \otimes \xi_{\text{ur}}^* : \mathcal{H}_{U_n}^- \rightarrow \mathcal{H}_{U_{\tilde{n}}}^-$. By Lemma 2.3, Lemma 4.2 and Lemma 4.6, we have an embedding of

$\mathcal{H}_{U_n}^-$ -modules

$$S_{U_{\vec{n}}, \kappa(W_{U_{\vec{n}}})}^{\text{cl}} \otimes \kappa(W_{U_{\vec{n}}}) \hookrightarrow \left(S_{U_n, \kappa(W_{U_n})}^{\text{cl}} \otimes \kappa(W_{U_n}) \right)^{\oplus C}$$

where $C \in \mathbf{N}$ is chosen as in Lemma 4.6. The result then follows. \square

4.2.5. *Existence of Ξ .*

Theorem 4.8. *Let $\sigma_\nu \in \mathfrak{S}_n$ be as in Lemma 4.2 for all $\nu \in S_p$, and let the data $(p, S, e_{U_{\vec{n}}}, \phi_{U_{\vec{n}}})$ and $(p, S, e_{U_n}, \phi_{U_n})$ be as in Section 4.2. Then the p -adic Langlands functoriality morphism*

$$\Xi : \mathcal{D}_{U_{\vec{n}}} \rightarrow \mathcal{D}_{U_n}$$

for the Langlands direct sum exists and is unique.

Proof. Define $\Xi = \Xi_2 \circ \Xi_1$ where Ξ_1 is the morphism appearing in Lemma 4.5 and Ξ_2 is the morphism appearing in Lemma 4.7. The result follows by Lemma 3.6 and Lemma 4.4. \square

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E-mail address: paul-james.white@all-souls.ox.ac.uk